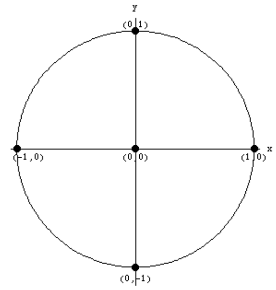
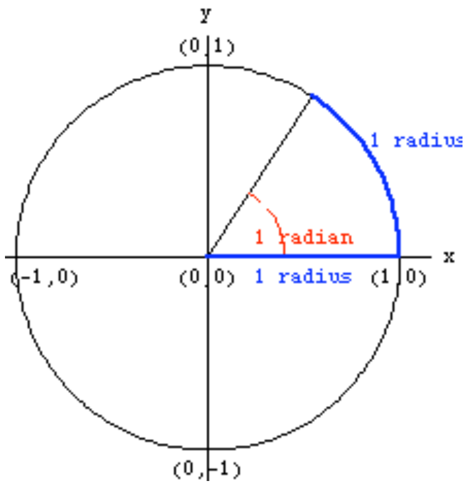
Math 1205 Trigonometry Review

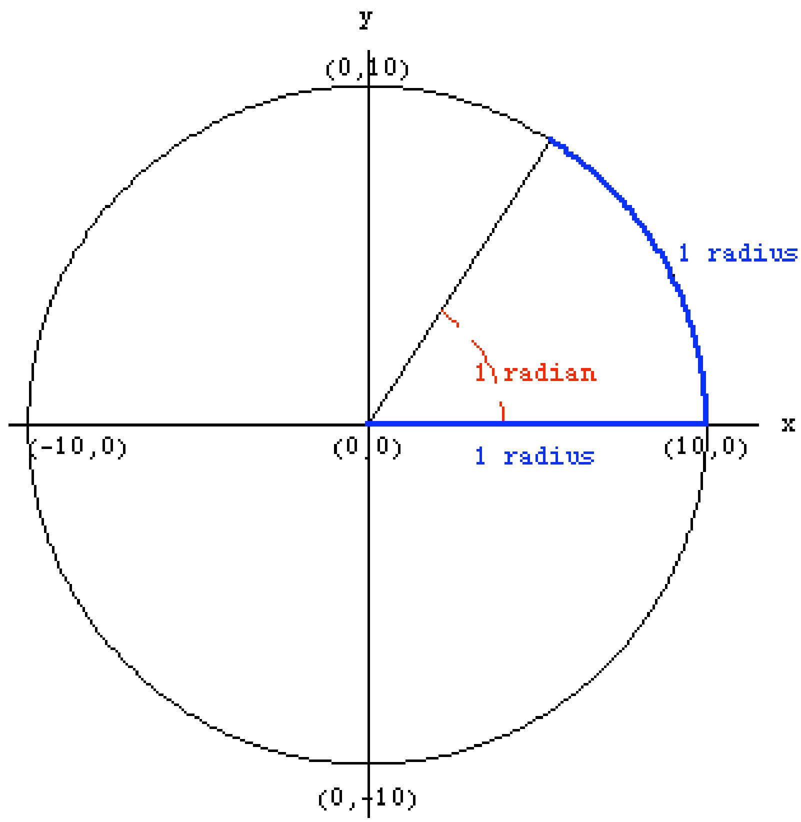
We begin with the unit circle.

The definition of a unit circle is: $x^2 + y^2 = 1$
where the center is $(0, 0)$ and the radius is 1.

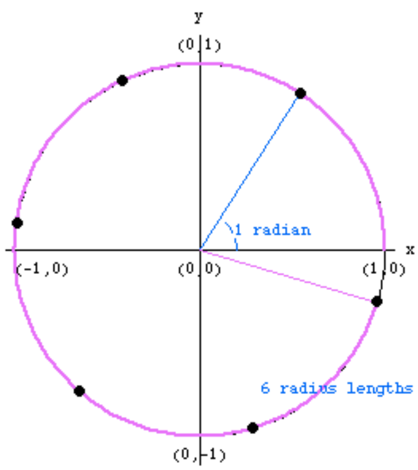


An angle of 1 radian is an angle at the center of a circle measured in the counterclockwise direction that subtends an arc length equal to 1 radius.
Notice that the angle does not change with the radius.



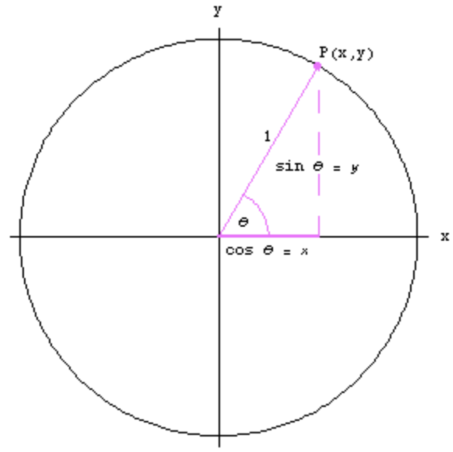


There are approximately 6 radius lengths around the circle.
That is, one complete turn around the circle is $2\pi \approx 6.28$ radians.



Define the Sine and Cosine functions:

Choose $P(x,y)$ a point on the unit circle where the terminal side of θ intersects with the circle.
Then $\cos\theta = x$ and $\sin\theta = y$.



We see that the Pythagorean Identity follows directly from these definitions:

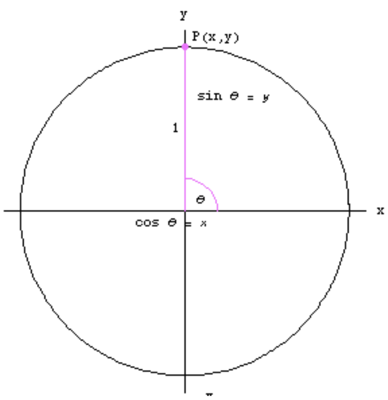
$$x^2 + y^2 = 1$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

$$\text{we know it as: } \sin^2\theta + \cos^2\theta = 1$$

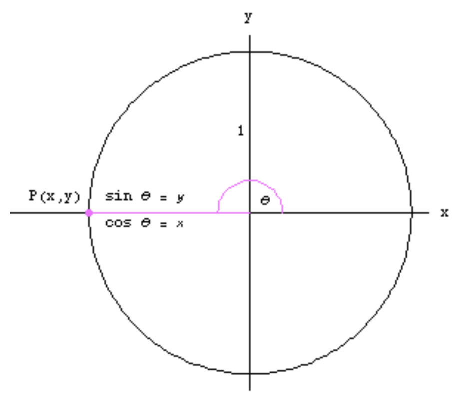
Example 1.
Determine:
$\sin(90°)$ and $\cos(90°)$

Recall that $90°$ corresponds to $\frac{\pi}{2}$ radians.



Example 2.
Determine:
$\sin(3\pi)$ and $\cos(3\pi)$

(How many degrees do 3π radians correspond to?)



We can read the answers from the graphs:

$$\sin(90°) = \sin\left(\frac{\pi}{2}\right) = y \text{ coordinate of } P = 1$$

$$\cos(90°) = \cos\left(\frac{\pi}{2}\right) = x \text{ coordinate of } P = 0$$

$$\sin(3\pi) = \sin(540°) = y \text{ coordinate of } P = 0$$

$$\cos(3\pi) = \cos(540°) = x \text{ coordinate of } P = -1$$

Problems 1 and 2:

1. Locate the following angles on a unit circle and find their sine and cosine.
   a. $-\frac{5\pi}{2}$ b. $\frac{5\pi}{2}$ c. $360°$ d. $-\pi$

2. Given: $\cos\theta = 0$ and $\sin\theta = 1$. Find the following:
   a. the smallest positive θ that satisfies the given equalities.
   b. one other θ that satisfies the given equalities.

   Note: θ_1 and θ_2 should be in radians.

There are six trigonometric functions. We have considered the sine and cosine functions. We can define the four remaining in terms of these functions.

The tangent function: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ The cotangent function: $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

The cosecant function: $\csc\theta = \dfrac{1}{\sin\theta}$ The secant function: $\sec\theta = \dfrac{1}{\cos\theta}$

We know all of the above functions will have points of discontinuity where the denominator is zero. The graphs of these functions all have vertical asymptotes at these points.

We will use the definition of the sine and cosine functions on the unit circle ($r = 1$) to find the sine and cosine for common reference angles.

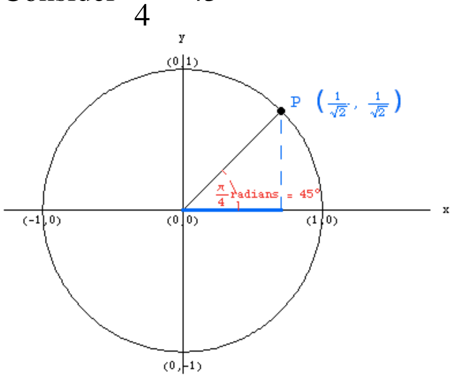
$\cos\theta = x$ and $\sin\theta = y$

We could use the sine and cosine graphs, however the unit circle is more useful for these problems.

The common angles that we are interested in are:

degrees	0	30	45	60	90	180	270	360
radians	0	π/6	π/4	π/3	π/2	π	3π/2	2π

Consider $\dfrac{\pi}{4} = 45°$



An angle of $\dfrac{\pi}{4}$ radians intersects the unit circle at the point,

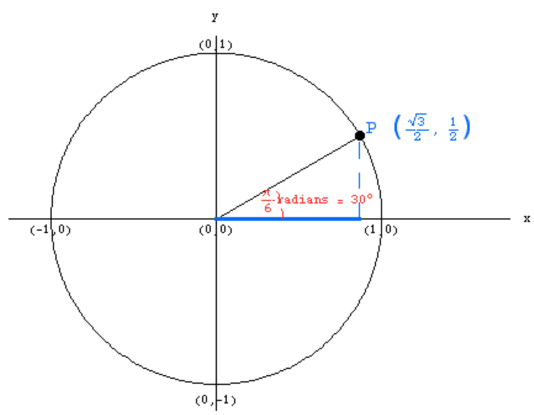
$P = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$.

Using the definition for sine and cosine, we have:

$\sin\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$ and $\cos\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$

Similarly, we find the sine and cosine of $\dfrac{\pi}{6} = 30°$:

$\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$ and $\cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$



We can complete the chart by working in the same manner to get:

degrees	0	30	45	60	90	180	270	360
radians	0	π/6	π/4	π/3	π/2	π	3π/2	2π
sin θ	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	-1	0
cos θ	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	-1	0	1

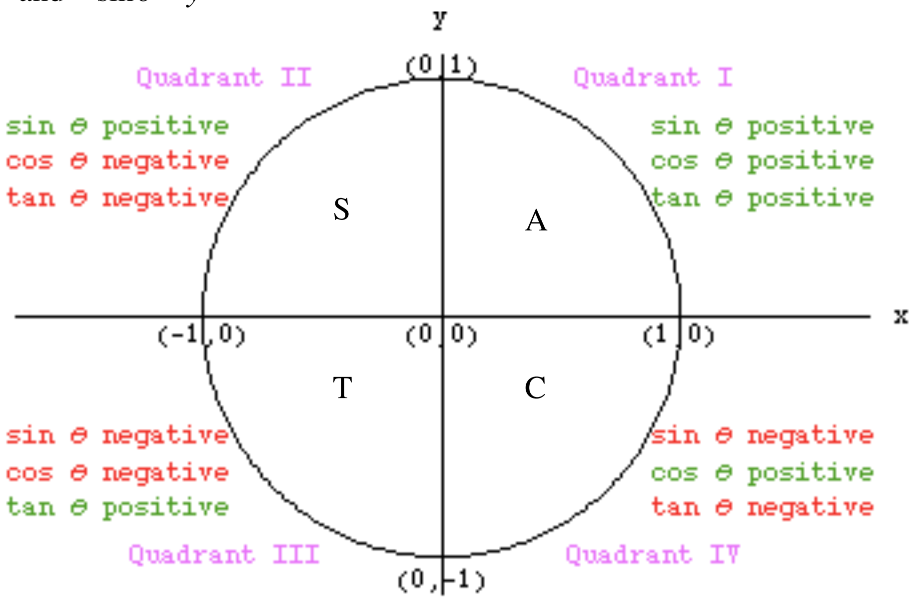
The common angles $< 90°$ listed above will become reference angles.

Problem 3.

Using the table, find: $\tan\frac{\pi}{6}$ and $\cot\frac{\pi}{4}$

Using the definition of the sine and cosine functions on the unit circle we can find the signs of the trigonometric functions in each quadrant.

$\cos\theta = x$ and $\sin\theta = y$



The above graph shows the results.

Problems 4 through 8:

4. In which of the four quadrants is the sine function positive?

5. In which of the four quadrants is the secant function negative?

6. In which of the four quadrants is the cosecant function positive and the cosine function negative?

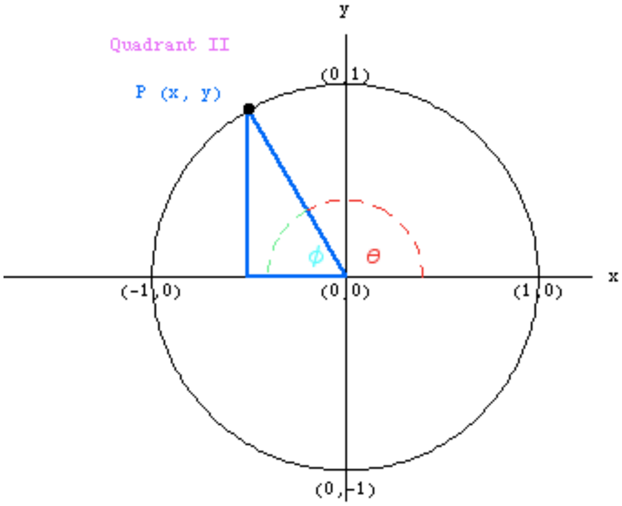
7. In which of the four quadrants do the tangent function and the cotangent function have the same signs?

8. Find the signs of $\sin\left(\frac{7\pi}{6}\right)$, $\cos\left(\frac{7\pi}{6}\right)$ and $\tan\left(\frac{7\pi}{6}\right)$.

Recall: $\cos\theta = x$ and $\sin\theta = y$
We have used the definitions to find the sine and cosine of common reference angles.
We have used the definitions to find the signs of the trigonometric functions in each quadrant.

We can now use these definitions to evaluate the trigonometric functions of multiples of common reference angles.

Example 3: Consider $\theta = \frac{2\pi}{3}$. θ is an angle in Quadrant II.
We will define a Reference Triangle.
A Reference Triangle is a Right Triangle formed by dropping a perpendicular line from the point, P, to the x axis. (Recall P is the point of intersection of the terminal side of θ and the unit circle.)



The blue triangle is The Reference Triangle. We call the acute angle at (0,0) within the triangle, ϕ, the reference angle.

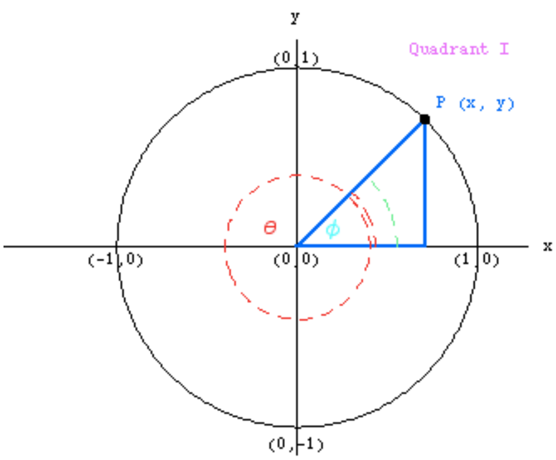
ϕ is closely related to θ:
The sine and cosine of θ have the same magnitude as the sine and cosine of ϕ. Only their signs may vary.

In this example, we "see" that $\phi = \pi - \theta = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$.

We know $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

We know the sine function is positive in Quadrant II and the cosine function is negative in Quadrant II.
Therefore: $\sin\left(\frac{2\pi}{3}\right) = +\frac{\sqrt{3}}{2}$ and $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

Example 4: Consider $\frac{9\pi}{4}$.
What quadrant is this in? How can I find out?
$\frac{9\pi}{4} = 2\pi + \frac{\pi}{4}$ = one complete revolution and $\frac{\pi}{4}$ more. $\therefore$ $\frac{9\pi}{4}$ is in Quadrant I.
The Reference Triangle is always a Right Triangle formed by dropping a perpendicular line from the point, P, to the x axis.



The blue triangle is The Reference Triangle. We always call the acute angle at (0,0) within the triangle, ϕ, the reference angle.

ϕ is always closely related to θ:
The sine and cosine of θ have the same magnitude as the sine and cosine of ϕ. Only their signs may vary.

In the above example, we "see" that $\phi = \theta - 2\pi = \frac{9\pi}{4} - 2\pi = \frac{\pi}{4}$.

We know $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

We know the sine function is positive in Quadrant I and the cosine function is positive in Quadrant I.

Therefore: $\sin\left(\frac{9\pi}{4}\right) = +\frac{1}{\sqrt{2}}$ and $\cos\left(\frac{9\pi}{4}\right) = +\frac{1}{\sqrt{2}}$.

Problem 9.
Use the method above to solve the following:

a. $\sin\left(-\frac{3\pi}{4}\right)$ b. $\cos\left(\frac{29\pi}{6}\right)$ c. $\tan(420°)$ d. $\sec\left(-\frac{9\pi}{4}\right)$

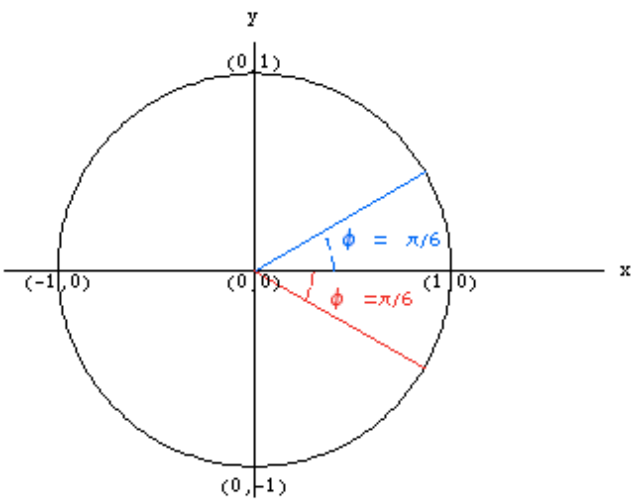
e. $\csc(510°)$ f. $\cot\left(\frac{19\pi}{3}\right)$

We can use this method to solve simple trigonometric equations.

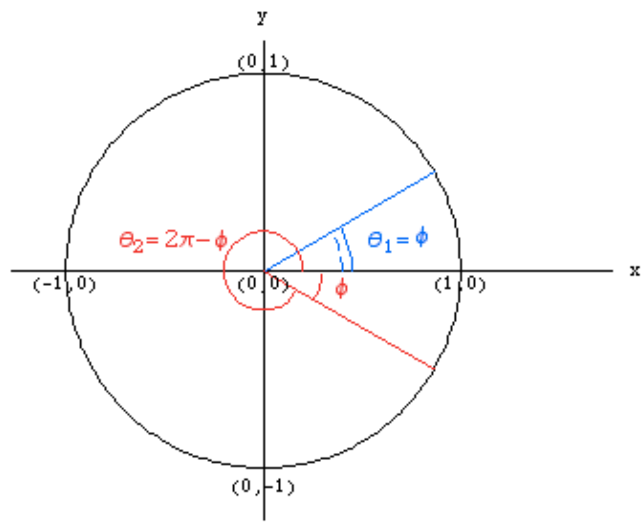
Example 5: Solve the following for θ where $0 \le \theta \le 2\pi$: $\cos\theta = \frac{\sqrt{3}}{2}$

We know the solution to: $\cos\phi = \frac{\sqrt{3}}{2}$ is $\phi = \frac{\pi}{6}$.
We also know that the cosine function is positive in quadrants I and IV.

Therefore our reference angle looks like this:



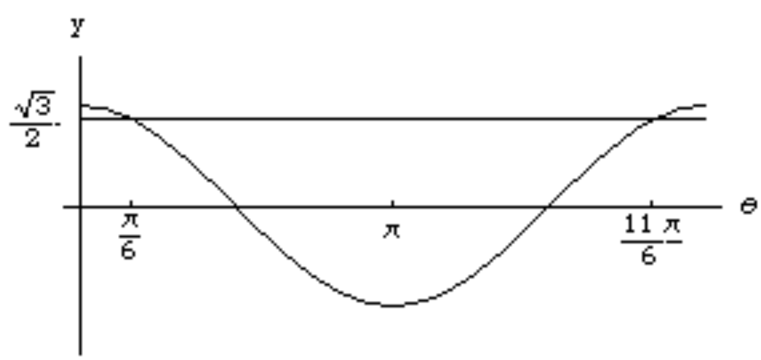
And our solutions look like this:



So:
$\theta_1 = \phi = \frac{\pi}{6}$ and $\theta_2 = 2\pi - \phi = 2\pi - \frac{\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6} = \frac{11\pi}{6}$

There is another method of solving: $\cos\theta = \frac{\sqrt{3}}{2}$.

We can graph $y = \cos\theta$ and $y = \frac{\sqrt{3}}{2}$ on the same set of axes and find their points of intersection.



We "see" the first point of intersection is: $\theta_1 = \frac{\pi}{6}$

and the second point of intersection is: $\theta_2 = \frac{11\pi}{6}$

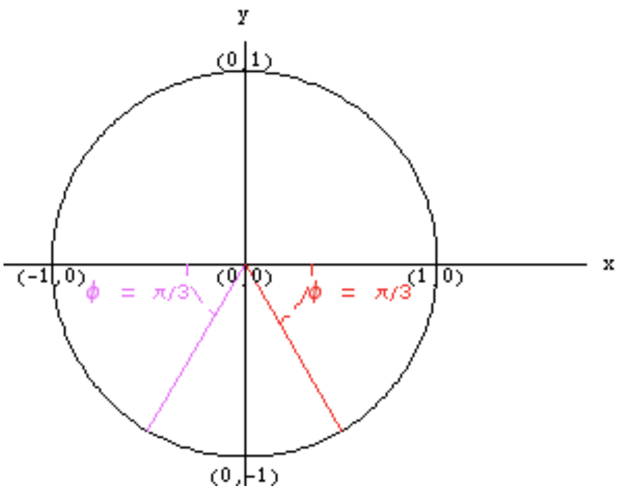
We will commonly use the first method as it is more useful for Calculus.

Example 6: Solve the following for θ where $0 \le \theta \le 2\pi$: $\sin\theta = -\frac{\sqrt{3}}{2}$

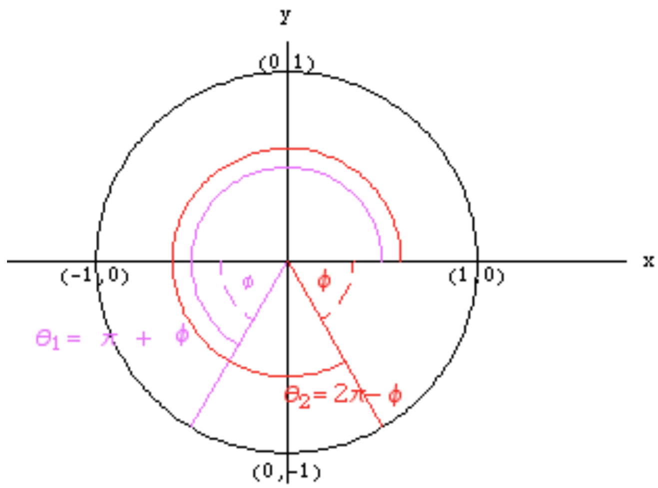
We know the solution to: $\sin\theta = +\frac{\sqrt{3}}{2}$ is $\phi = \frac{\pi}{3}$.

We also know that the sine function is negative in quadrants III and IV.

Therefore our reference angle looks like this:



And our solutions look like this:



So: $\theta_1 = \pi + \phi = \frac{4\pi}{3}$ and $\theta_2 = 2\pi - \phi = \frac{5\pi}{3}$

NOTE: Never solve this and similar problems by plugging a **negative** number into your calculator.
With the sine function, you will get a negative angle (IV quadrant on the unit circle).
With the cosine function, you will get an angle in quadrant II only.
Your calculator is set up to find the inverse trigonometric functions. This is **NOT** what we want in these problems.

To **summarize this method** of solving simple trigonometric equations:

1. Locate ϕ a small **positive** angle between 0 and $\frac{\pi}{2}$.
2. Place ϕ in the quadrants corresponding to the given equation.
3. Find a θ in the appropriate quadrants.

Problem 10. Given: $\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta$ is negative.

Find: the quadrant of θ and $\sec\theta$.

Problem 11. Solve for θ where $0 \le \theta \le 2\pi$ in the following problems.

a. $\cos\theta = -\frac{1}{\sqrt{2}}$ b. $\tan\theta = -\sqrt{3}$ c. $\sin\theta = \frac{1}{2}$

Example 7: In Example 6, Solve the following for θ where $0 \le \theta < 2\pi$: $\sin\theta = -\frac{\sqrt{3}}{2}$, we found $\theta_1 = \frac{4\pi}{3}$ and $\theta_2 = \frac{5\pi}{3}$. How does our answer change if the question asks us to solve for all θ?

We know that the sine function is periodic. It repeats every 2π radians. Then our solutions should also repeat every 2π radians. Our solutions become:

$$\theta_1 = \frac{4\pi}{3},\ \frac{4\pi}{3} \pm 2\pi,\ \frac{4\pi}{3} \pm (2)2\pi,\ \frac{4\pi}{3} \pm (3)2\pi\ \ . . .$$

$\therefore$ in general $\theta_1 = \frac{4\pi}{3} + (n)2\pi$ where n is an integer

and

$$\theta_2 = \frac{5\pi}{3},\ \frac{5\pi}{3} \pm 2\pi,\ \frac{5\pi}{3} \pm (2)2\pi,\ \frac{5\pi}{3} \pm (3)2\pi\ \ . . .$$

$\therefore$ in general $\theta_2 = \frac{5\pi}{3} + (n)2\pi$ where n is an integer

Problem 12. Find all the values for θ from Example 5, $\cos\theta = \frac{\sqrt{3}}{2}$

Example 8: Solve $\cos 2\pi x = 0$ for x in $(-1,1)$.

We know $\cos\phi = 0$ when $\phi = \pm\frac{\pi}{2}$. $\therefore$ $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}\ . . . \frac{n\pi}{2}$ where n is an odd integer.

So $2\pi x = \frac{n\pi}{2} \Rightarrow x = \frac{n}{4}$ where n is an odd integer. And for x in $(-1,1)$ the solution is $x = \pm\frac{1}{4}, \pm\frac{3}{4}$.

Problem 13. Solve the following for x on the given intervals. Use the methods described above.

a. $\sin 2\pi x = 0$ for x in $(0,2)$

b. $\tan\frac{\pi}{2}x = -\frac{1}{\sqrt{3}}$ for x in $(1,4)$

c. $\cos\frac{\pi}{3}x = 1$ for x in $(-7,7)$

Identities:

$$\sin^2 x + \cos^2 x = 1$$

divide by $\cos^2 x$: $\tan^2 x + 1 = \sec^2 x$

divide by $\sin^2 x$: $1 + \cot^2 x = \csc^2 x$

Double Angle Formulas:

$$\sin 2x = 2\sin x\cos x \qquad \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

Half – Angle Formulas:

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$