Section 9.2 The Unit Circle

Definition: The **Unit Circle** is a circle with radius 1.
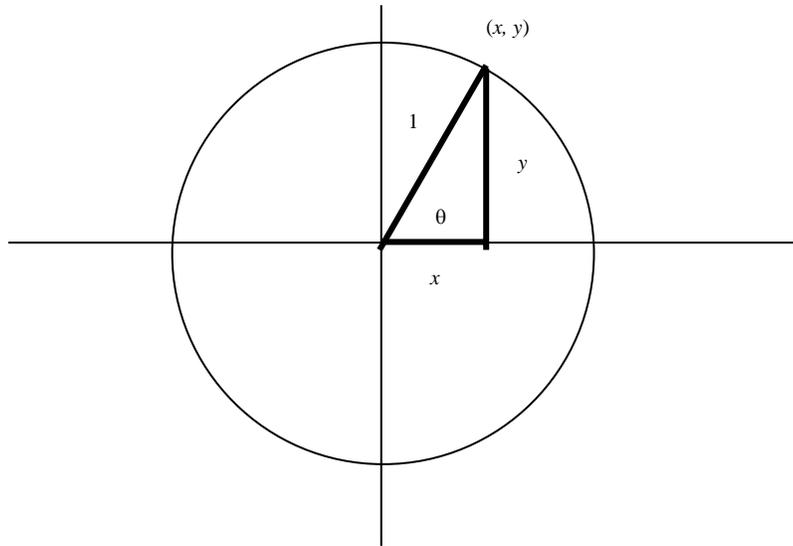
Every point (x, y) on the unit circle corresponds to some angle θ. For example:

Point (x, y)	Angle θ
(1, 0)	0 or 2π
(0, 1)	90 or $\pi/2$
(-1, 0)	180 or π
(0, -1)	270 or $3\pi/2$

Question: How does this affect us?

Answer: We can define trigonometric functions based on the coordinates of the point on the unit circle which corresponds to the angle. Notice that since the circle has radius 1 the triangle in bold in the figure above has hypotenuse 1, height length y and base length x. We can now use the techniques from section 9.4 to define the six trigonometric functions:

$$\sin\theta = y \qquad\qquad \csc\theta = \frac{1}{\sin\theta} = \frac{1}{y}$$

$$\cos\theta = x \qquad\qquad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad\qquad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$$

And every point on the circle is $(x, y) = (\cos\theta, \sin\theta)$

We can use the two special triangles we looked at in section 9.4 to fill in the unit circle for many "standard" angles. In the following diagram, each point on the unit circle is labeled first with its coordinates (exact values), then with the angle in degrees, then with the angle in radians. Points in the lower hemisphere have both positive and negative angles marked.
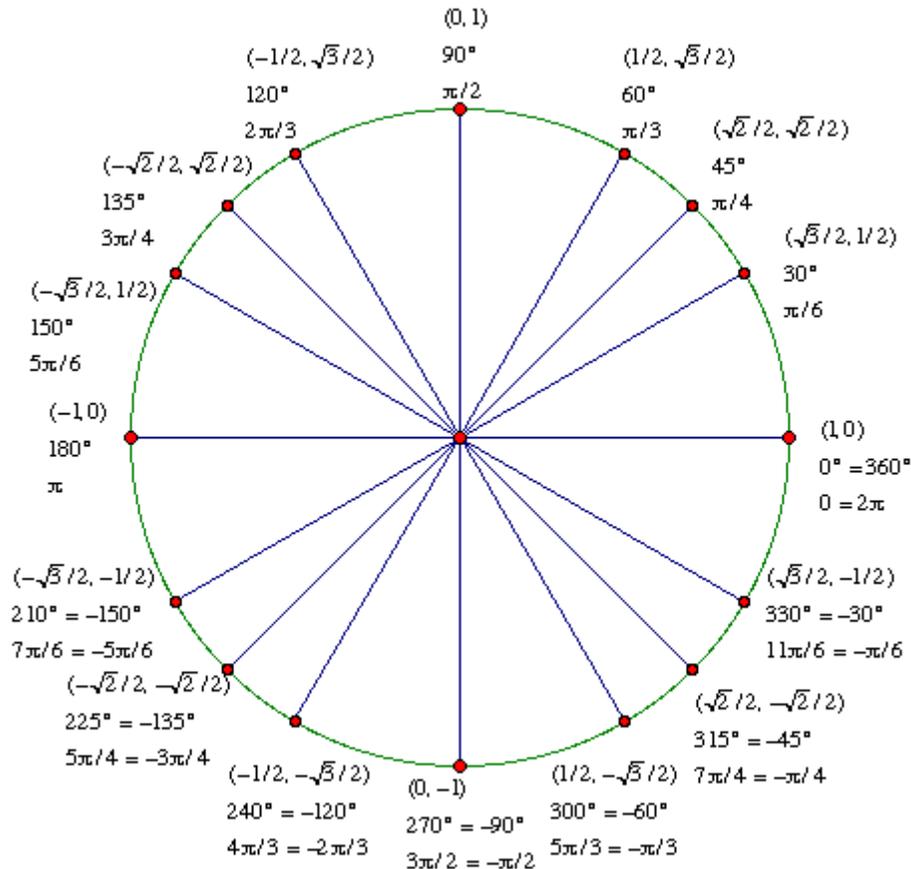
For any trigonometry problem you can either use the unit circle or the triangle techniques in section 9.4. If you want to use the unit circle on the test you must have it memorized.

Ex 1: Find the six trigonometric functions for the following angles:

a) $\vartheta = -\dfrac{2\pi}{3}$. θ corresponds to the point $\left(-1/2, -\sqrt{3}/2\right) = (\cos\theta, \sin\theta)$

$$\sin\theta = -\sqrt{3}/2 \qquad\qquad \csc\theta = \dfrac{1}{\sin\theta} = -2/\sqrt{3}$$

$$\cos\theta = -1/2 \qquad\qquad \sec\theta = \dfrac{1}{\cos\theta} = -2$$

$$\tan\theta = \dfrac{(-\sqrt{3}/2)}{(-1/2)} = \sqrt{3} \qquad\qquad \cot\theta = \dfrac{1}{\tan\theta} = \dfrac{x}{y} = 1/\sqrt{3}$$

b) $\theta = 135° = \dfrac{3\pi}{4}$. θ corresponds to the point $\left(-\sqrt{2}/2, \sqrt{2}/2\right) = (\cos\theta, \sin\theta)$

$$\sin\theta = \sqrt{2}/2 \qquad\qquad \csc\theta = \dfrac{1}{\sin\theta} = \sqrt{2}$$

$$\cos\theta = -\sqrt{2}/2 \qquad\qquad \sec\theta = \dfrac{1}{\cos\theta} = -\sqrt{2}$$

$$\tan\theta = \dfrac{(\sqrt{2}/2)}{(-\sqrt{2}/2)} = -1 \qquad\qquad \cot\theta = \dfrac{1}{\tan\theta} = \dfrac{x}{y} = -1$$

c) $\theta = \pi = 180°$. θ corresponds to the point $(-1, 0) = (\cos\theta, \sin\theta)$

$$\sin\theta = 0 \qquad\qquad \csc\theta = \dfrac{1}{\sin\theta} = Undefined$$

$$\cos\theta = -1 \qquad\qquad \sec\theta = \dfrac{1}{\cos\theta} = -1$$

$$\tan\theta = \dfrac{0}{-1} = 0 \qquad\qquad \cot\theta = \dfrac{1}{\tan\theta} = undefined$$

Note: We can not divide by zero so cosecant and cotangent are both *undefined*.

d) $\vartheta = 270° = \dfrac{3\pi}{2}$. θ corresponds to the point $(0, -1) = (\cos\theta, \sin\theta)$

$$\sin\theta = \qquad\qquad \csc\theta = \dfrac{1}{\sin\theta} =$$

$$\cos\theta = \qquad\qquad \sec\theta = \dfrac{1}{\cos\theta} =$$

$$\tan\theta = \qquad\qquad \cot\theta = \dfrac{1}{\tan\theta} =$$

Domain and Period of sine and cosine:

Once again you will notice that by adding or subtracting 360° or 2π to any angle you get back to the same angle on the graph (coterminal). So the following relationship is true:

$$\sin\vartheta = (\pi/3) = \sin(\pi/3 + 2\pi).$$

Functions that repeat values at a regular interval are called **Periodic**.

$$f(x) = f(x + c) \text{ for some } c.$$

In the case of sine and cosine we have the following relationships:

$\sin(\theta + 2n\pi) = \sin\theta$ and $\cos(\theta + 2n\pi) = \cos\theta$ where n can be any integer.

We also have the following relationships:

$$\cos(-t) = \cos(t) \qquad \text{cosine is an even function.}$$
$$\sin(-t) = -\sin(t) \qquad \text{sine is an odd function.}$$

Ex 2: Suppose $\cos(t) = -3/4$. Find the following:

a) $\cos(-t) =$

b) $\sec(-t) =$

Ex 3: Find $\cos(5\pi)$

5π is larger than 2π (one time around the circle) so we need to find a coterminal angle between 0 and 2π. To do this we subtract 2π until the angle is where we want it.

5π - 2π - $2\pi = \pi$ So $\boxed{\cos(5\pi) = \cos \pi = -1}$

Ex 4: Find $\sin(-9\pi/4)$

-9π/4 is not between 0 and 2π (one time around the circle) so we need to find a coterminal angle between 0 and 2π. To do this we add 2π until the angle is where we want it.

-9π/4 + 2π +2π = 7π/4   So $\boxed{\sin(-9\pi/4) = \sin(7\pi/4) = -\sqrt{2}/2}$ (From the unit circle).