

Elementary Functions

Part 4, Trigonometry
Lecture 4.1a, The Unit Circle

Dr. Ken W. Smith

Sam Houston State University

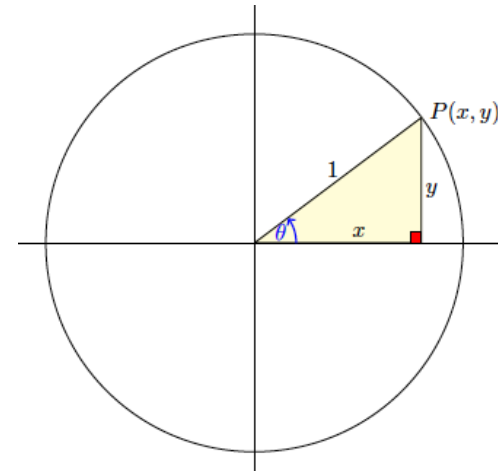
2013

The Unit Circle

Many important elementary functions involve computations on the unit circle.

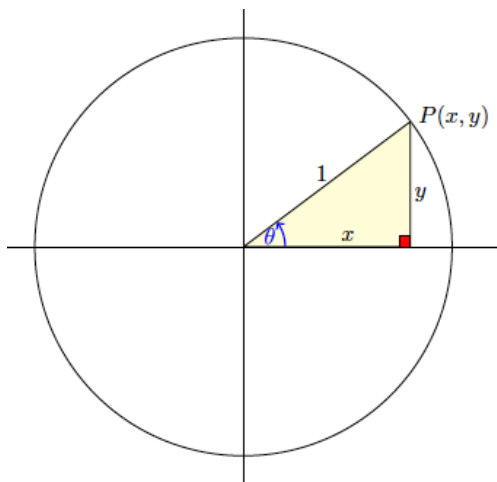
These “circular functions” are called by a different name, “trigonometric functions.”

But the best way to view them is as functions on the circle.



The Unit Circle

The **unit circle** is the circle centered at the origin $(0, 0)$ with radius 1. Draw a ray from the center of the circle out to a point $P(x, y)$ on the circle to create a **central angle** θ (drawn in blue, below.)

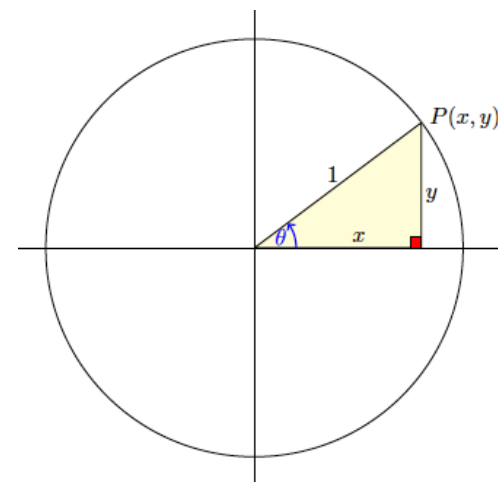


The Unit Circle

The radius of the circle is one, so $P(x, y)$ is a vertex of a right triangle with sides x and y and hypotenuse 1.

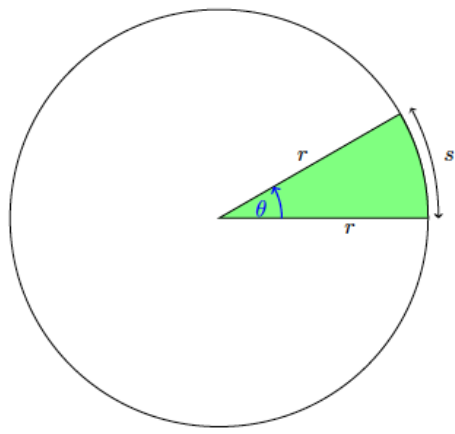
By the Pythagorean theorem, $P(x, y)$ solves the equation

$$x^2 + y^2 = 1 \tag{1}$$



An **arc** of the circle corresponds to a **central angle** created by drawing line segments from the endpoints of the arc to the center.

The Babylonians (4000 years ago!) divided the circle into 360 pieces, called **degrees**. This choice is a very *human* one; it does *not* have a natural mathematical reason. (It is not “intrinsic” to the circle.)



The most natural way to measure arcs on a circle is by the intrinsic unit of measurement which comes with the circle, that is, the length of the radius.

The unit of length given by the radius is called a **radian**; we will measure arcs and their angles by radians.

This sometime involves the number π .

The ancient Greeks noticed that the circumference C of a circle was always slightly more than three times the diameter d of the circle. They used the letter π to denote this ratio, so that

$$\pi := \frac{C}{d} = \frac{C}{2r}.$$

Since the length d of the diameter of a circle is merely twice the radius r then this is often expressed in the equation

$$C = 2\pi r. \tag{2}$$

The circumference of a circle, that is, the arc going completely around the circle once, is an arc of 360 degrees, and so the correspondence between the ancient Babylonian measurement of degrees and the natural measurement of radians is

$$360 \text{ degrees} = 2\pi \text{ radians}$$

or, after dividing by 2,

$$\boxed{180 \text{ degrees} = \pi \text{ radians}}$$

We can write this equation as a “conversion factor”, that is,

$$\frac{\pi \text{ radians}}{180^\circ} = 1 \tag{3}$$

and so if we want to convert degrees to radians, we multiply by this factor.

Here are some sample problems based on these unit circle terms.

1 Change 240° to radians

Solution. Since π radians represents 180° (halfway around the circle) then $240^\circ = \frac{240}{180}\pi$ radians, which is equal to $\frac{4\pi}{3}$ radians.

2 Change 40° to radians.

Solution. Mechanically we may multiply 40° by the conversion factor $\frac{\pi \text{ radians}}{180^\circ}$, so that degrees cancel out:

$$40^\circ = 40^\circ \left(\frac{\pi \text{ radians}}{180^\circ} \right) = \frac{40}{180}\pi \text{ radians} = \frac{2}{9}\pi \text{ radians}.$$

3 Change 1.5 radians to degrees.

Solution. Mechanically we multiply 1.5 radians by $\frac{180^\circ}{\pi \text{ radians}}$ (the reciprocal of the earlier conversion factor) so that radians cancel and the answer is in degrees:

$$1.5 \text{ radians} = 1.5 \text{ radians} \left(\frac{180^\circ}{\pi \text{ radians}} \right) = \left(\frac{1.5 \cdot 180}{\pi} \right)^\circ = \left(\frac{270}{\pi} \right)^\circ.$$

In the next presentation, we will look at arclength.

(End)

Arc length and sector area

Working with **radians** instead of **degrees** simplifies most computations involving a circle. In this presentation we look first at arc length problems on an arbitrary circle and then at areas of sectors of an arbitrary circle.

As we go through this material, notice how important radians are to our computations!

Arc length

If we measure our angles θ in radians then the relationship between the central angle θ and the length s of the corresponding arc is an easy one.

The length s should be measured *in the same units* as the radius and so if θ is measured in radians, we just need to write the radius r in these same units.

Elementary Functions

Part 4, Trigonometry
Lecture 4.1b, Arc Length

Dr. Ken W. Smith

Sam Houston State University

2013

For example, suppose the central angle θ is 2 radians and the circle has radius is 20 miles.

Then the length s of the corresponding arc is

$$2 \text{ radians} = 2 \cdot 20 \text{ miles} = 40 \text{ miles.}$$

We can state this relationship as an equation:

$$s = \theta r$$

but it should be obvious.

(If there are 3 feet in a yard, how many feet are there in 8 yards? Obviously $8 \cdot \text{yards} = 8 \cdot 3 \text{ feet} = 24 \text{ feet}$. There is no difference between this change of units computation, yards-to-feet, and the change of units computation for arc length.)

- 1 Find the arclength s if the radius of the circle is 20 feet and the arc marks out an angle of 3 radians.

Solution. If the radius is 20 feet and the arc subtends an angle of 3 radians then 3 radians is equal to $(3)(20) = 60 \text{ feet}$.

- 2 Find the length s of the arc of the circle if the arc is subtended by the angle $\frac{\pi}{12}$ radians and the radius of the circle is 24 meters.

Solution. $\frac{\pi}{12}$ radians is $\frac{\pi}{12} \cdot 24 \text{ m} = 2\pi \text{ meters}$.

- 3 Find the arclength s if the radius of the circle is 20 feet and the arc marks out an angle of 10° .

Solution. 10° is equal to $\frac{\pi}{18}$ radians. (We have to ALWAYS work in radians here!) If the radius is 20 feet and the arc subtends an angle of $\frac{\pi}{18}$ radians then $\frac{\pi}{18}$ radians is equal to $(\frac{\pi}{18})(20) = \frac{20\pi}{18} = \frac{10\pi}{9} \text{ feet}$.

An example from history.

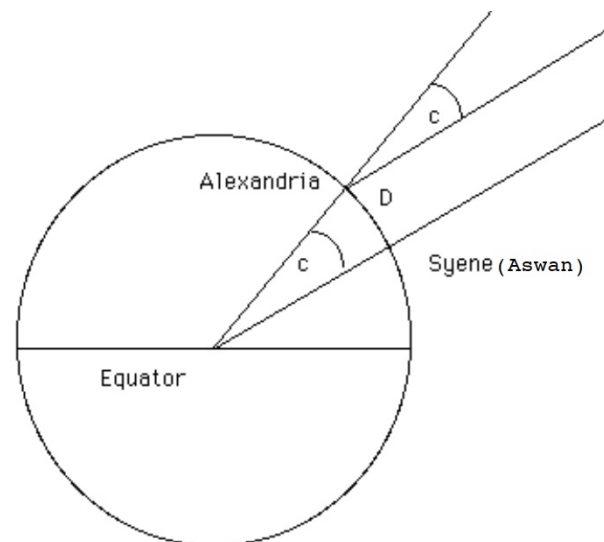
The ancient Greeks believed, correctly, that the earth was a sphere.

(They recognized a lunar eclipse as the shadow of the earth moving across the moon and noticed that the shadow was always part of a circle so they reasoned that the earth, like the moon, was a large ball.)

About 230 BC, a Greek scientist, [Eratosthenes](#), attempted to determine the radius of the earth. He noted that on the day of the summer solstice (around June 21) the sun was directly overhead in the city of Aswan (then called Syrene). But 500 miles due north, in the city of Alexandria where he lived, the sun was 7.5° south of overhead.

An example from history.

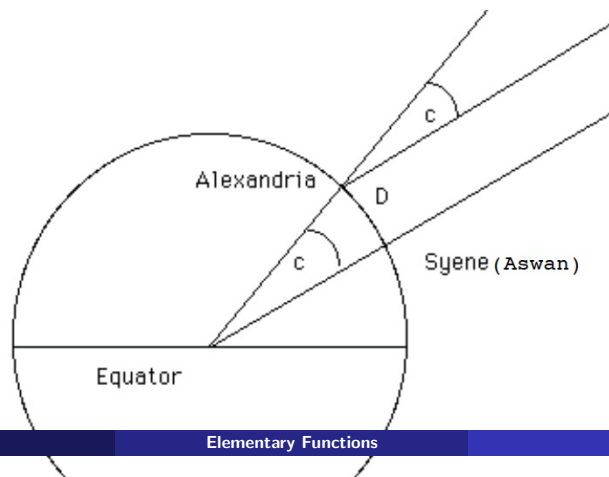
Assuming that the earth was a perfect sphere and so the cities Aswan (Syene) and Alexandria lie on a great circle of the earth, Eratosthenes was able to calculate the radius of the earth. Here is how.



The central angle $C = 7.5^\circ$ is equal to about $\pi/24$ radians.

Since 500 miles equals $\pi/24$ radians then $500 = (\pi/24)r$ and so $r = 12000/\pi$ which is about 3820 miles.

Since the true radius of the earth is now known to be about 3960 miles, this is a rather accurate measurement of the size of the earth!



In the next presentation, we will look at other computations involving the central angle, angular speed and sector area.

(End)

Central Angles and Arcs

Elementary Functions

Part 4, Trigonometry
Lecture 4.1c, More On The Unit Circle

Dr. Ken W. Smith

Sam Houston State University

2013

Linear speed v. angular speed

Our earlier “obvious” equation $s = r\theta$, relating arc to angle, also works with measurements of speed. The angular speed of a spinning object is measured in radians per unit of time. The linear speed is the speed a particle on the spinning circle, measure in linear units (feet, meters) per unit of time.

Suppose a merry-go-round is spinning at 6 revolutions per minute. The radius of the merry-go-round is 30 feet. How fast is someone traveling if they are standing at the edge of the merry-go-round?

Solution. Six revolutions per minute is $6 \cdot 2\pi = 12\pi$ radians per minute. This is the angular speed of the merry-go-round. In this problem, a radian is 30 feet so 12π radians per minute is $(12\pi)(30 \text{ feet})$ per minute $= \text{360}\pi \text{ ft/min} \approx 1131 \text{ ft/min}$. (This is about 13 miles per hour.)

Given an angle θ we will talk about the smallest angle made by the arc and the x -axis. For example, an angle of 150° has **reference angle** 30° degrees since the ray from the origin to the circle makes a 30° angle with the x -axis.

The reference angle for $\theta = 210^\circ$ is also a 30° angle in the third quadrant.

Draw, in standard position, the angle whose measurement, in radians, is $\frac{4\pi}{3}$. What is the reference angle for $\frac{4\pi}{3}$?

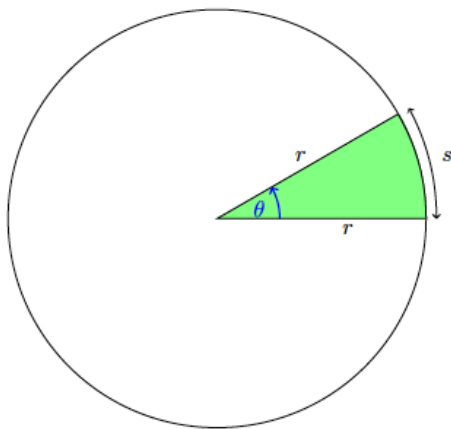
Solution. This will be the angle 240° , dropping into the third quadrant, 60° below the x -axis. The reference angle is the smallest angle created with one side on the x -axis; here it is 60° or $\pi/3$ radians.

Area of a sector

If we join the endpoints of an arc to the center of the circle, we obtain a region often shaped somewhat like a triangle.

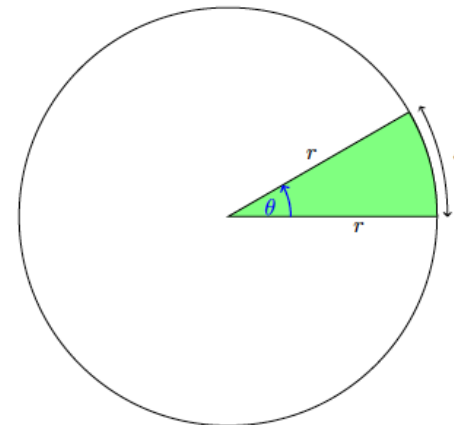
This **sector** is bounded by two radii and the arc.

A typical slice of pizza is an example of a sector of a circle.



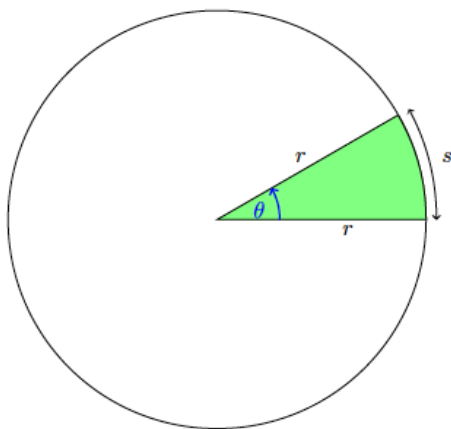
Area of a sector

The area of a sector can be determined by acting as if the sector were a triangle with **base** equal to the arc length and **height** equal to the radius.



Area of a sector

Since the area of a triangle is one-half the product of the **base** and the **height** then the area of a sector is one-half the product of the **arc length** and the **radius**,



Area of a sector

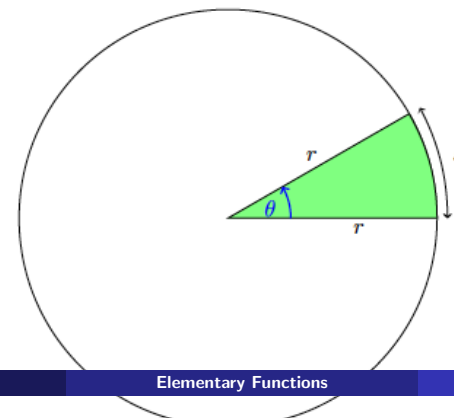
So one equation for the area of a sector is

$$A = \frac{1}{2}rs$$

We can substitute $r\theta$ for s and so

$$A = \frac{1}{2}r^2\theta$$

where θ is (of course!) measured in radians.



A worked problem.

Find the area A of a sector if the radius is 20 feet and the arc marks out an angle of 3 radians.

Solution. The area of a sector satisfies the equation $A = \frac{1}{2}rs$. In an earlier problem we computed $s = (20)(3)$ feet = 60 feet so

$$A = \frac{1}{2}(20)(60) = \boxed{600 \text{ square feet}}.$$

Central Angles and Arcs

In the next presentation, we will look at the two most important trig functions,

cosine and **sine**.

(End)

Elementary Functions

Part 4, Trigonometry
Lecture 4.1d, Cosine and Sine

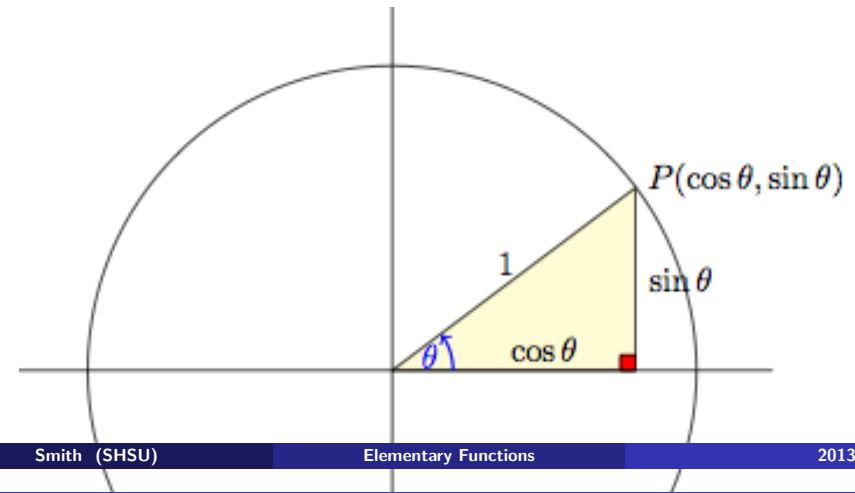
Dr. Ken W. Smith

Sam Houston State University

2013

The points $P(x, y)$ on the unit circle provide us two obvious “circular functions.” Given a point $P(x, y)$, draw the central angle with arc passing from $(1, 0)$ to (x, y) , passing **counterclockwise** around the circle. The point $P(x, y)$ depends on the central angle θ .

Traditionally the x -value of the point P is the **cosine** of θ and the y -value of P is the **sine** of θ .



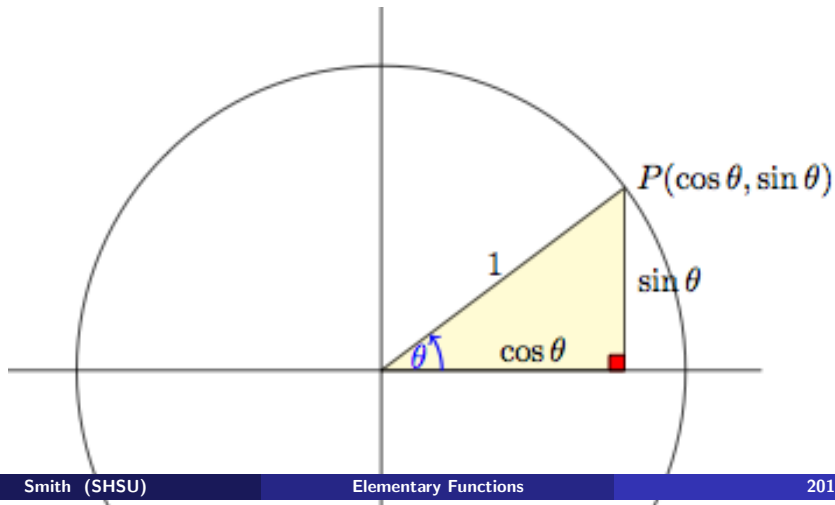
Sine and cosine functions on the unit circle

Sine and cosine functions on the unit circle

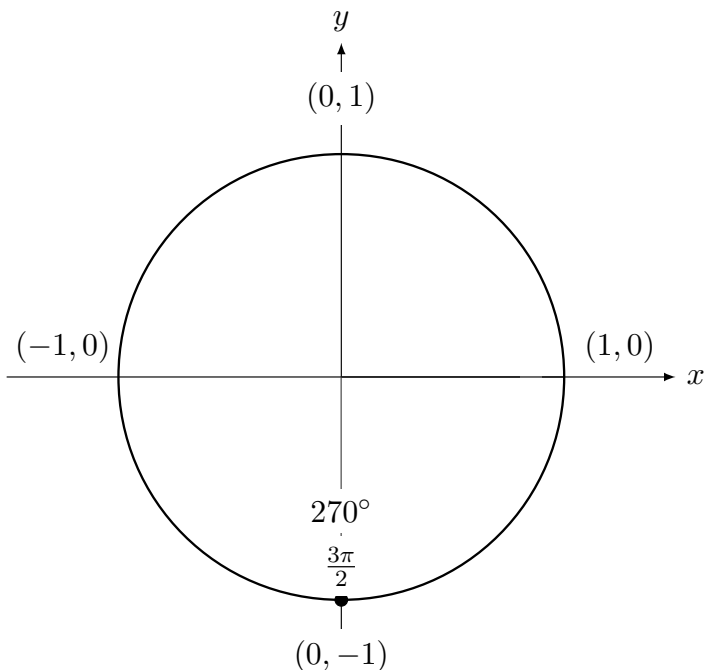
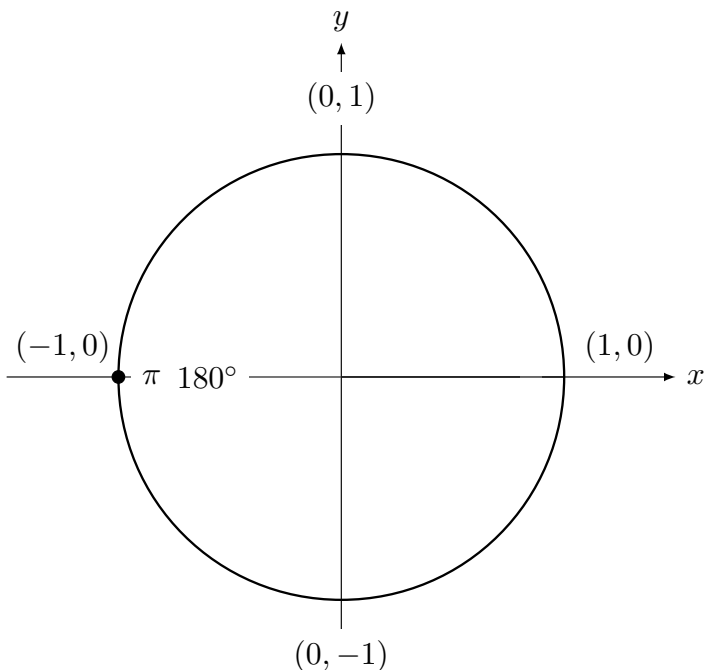
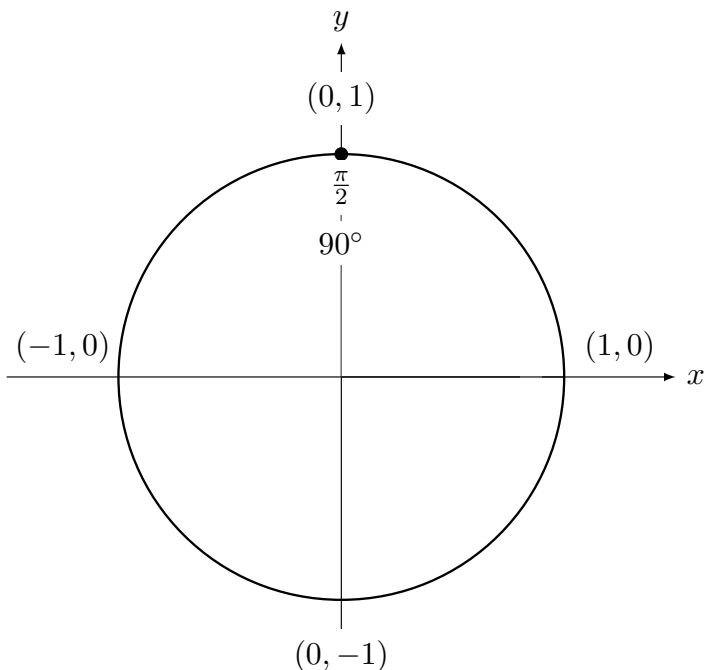
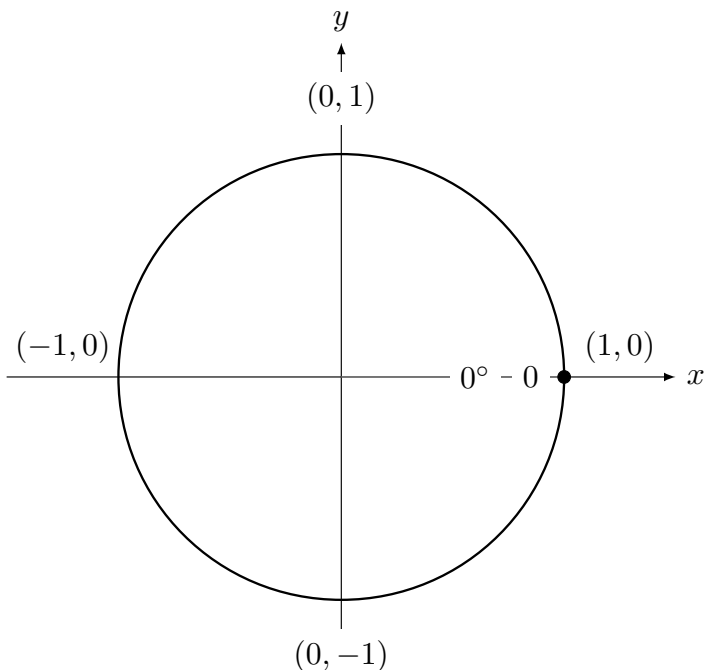
We abbreviate these functions by writing “cos” for cosine and “sin” for sine.

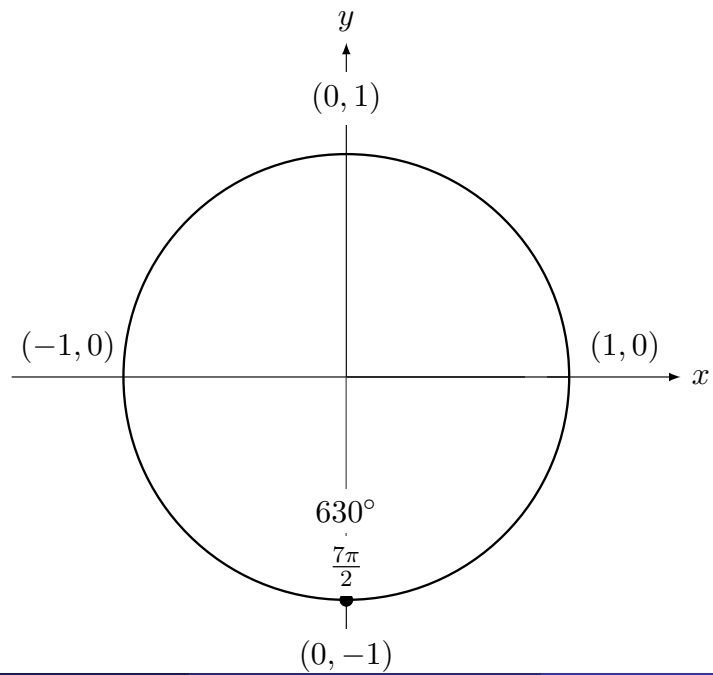
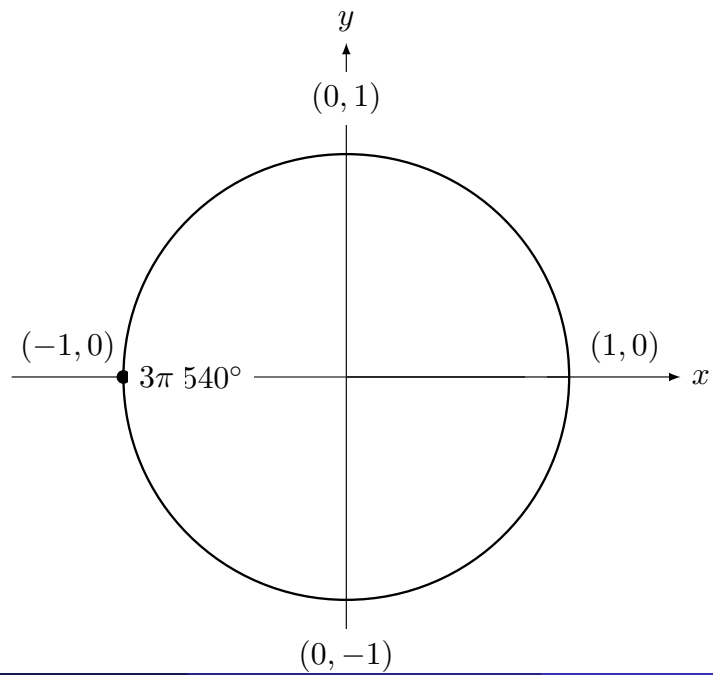
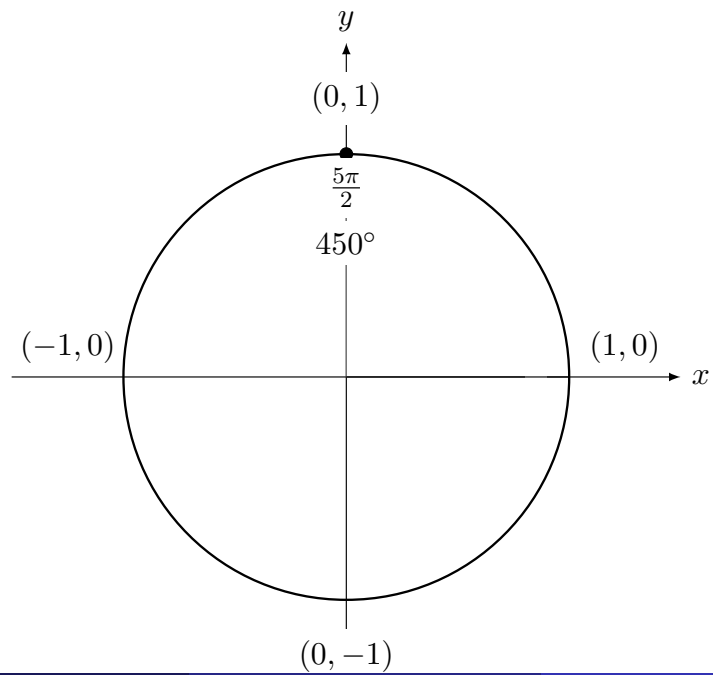
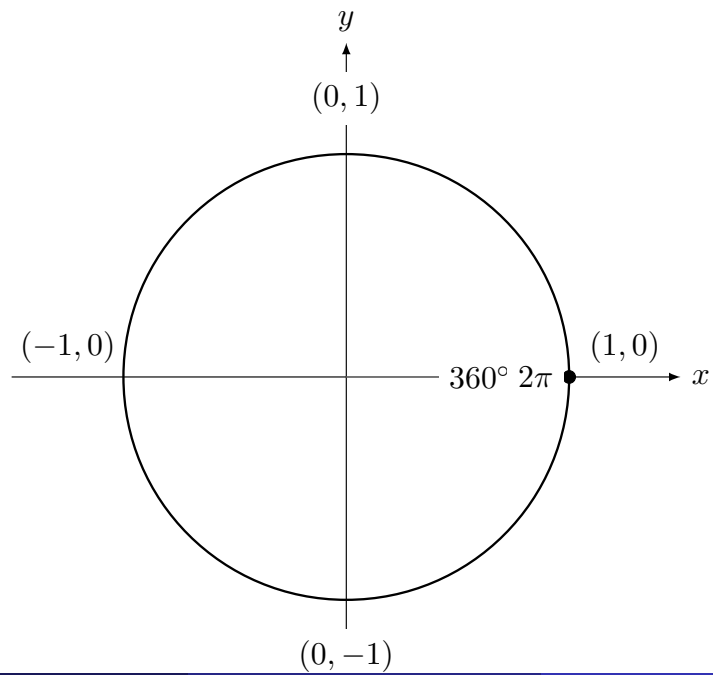
Remember that on the unit circle $x = \cos(\theta)$ and $y = \sin(\theta)$

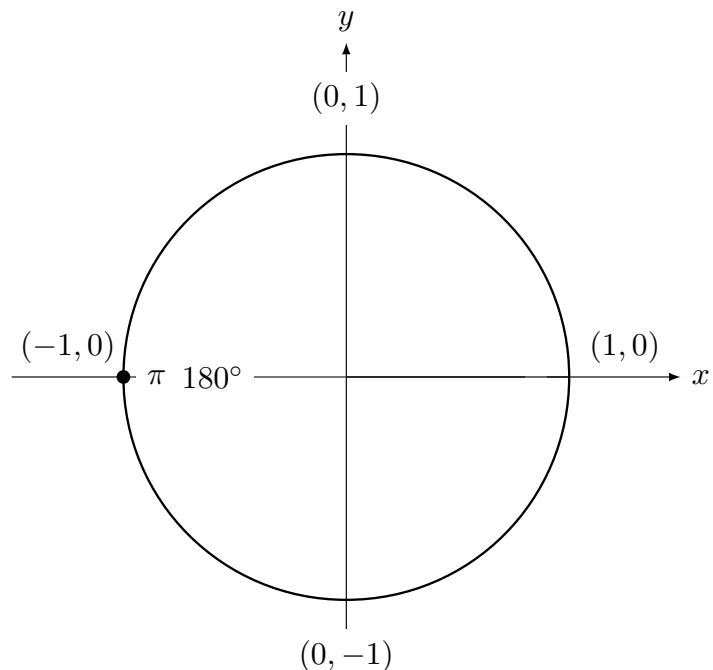
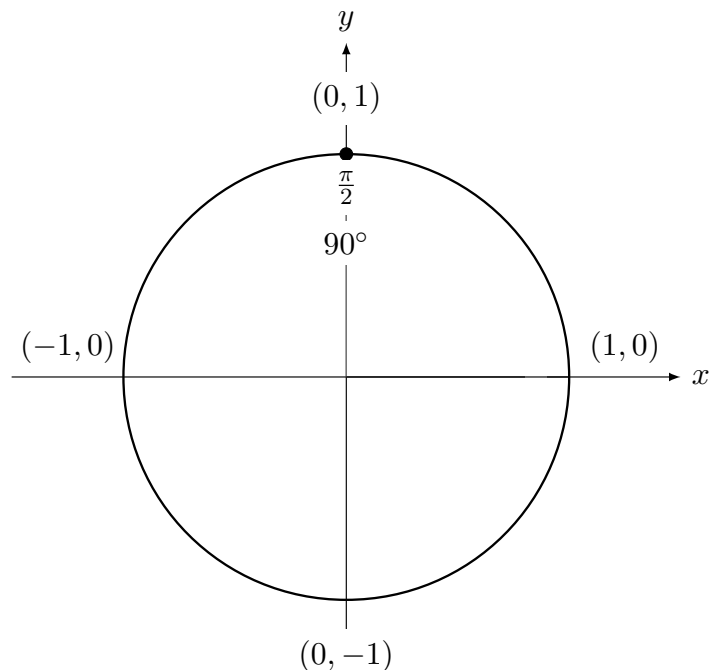
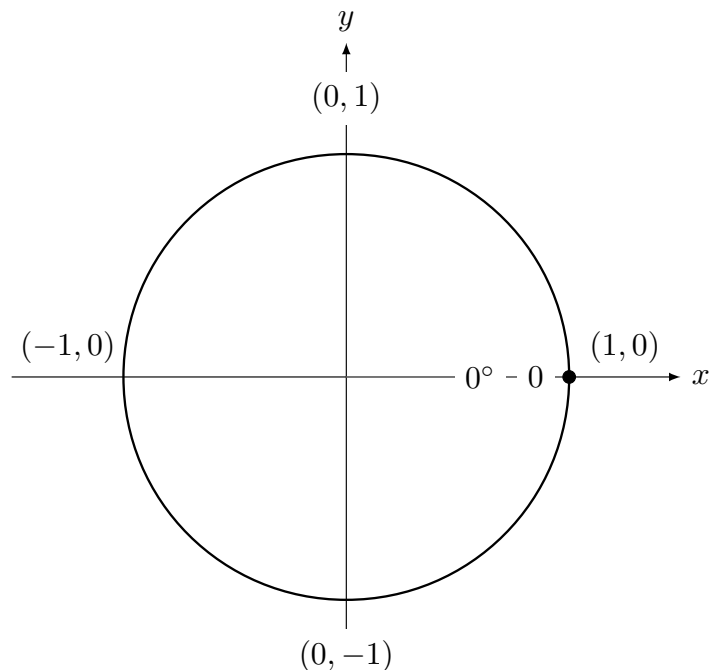
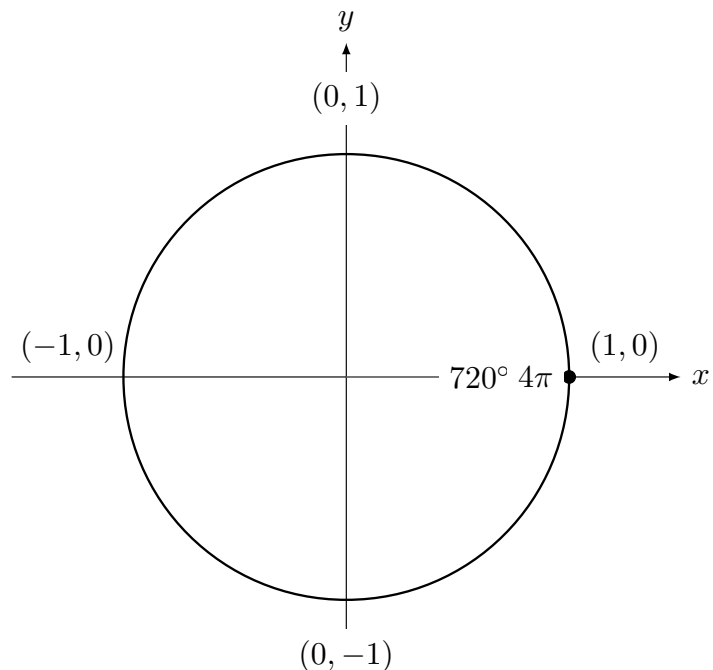
(Notice that these functions are in alphabetical order: since x comes before y in the alphabet, “cosine” comes before “sine” in the alphabet!)



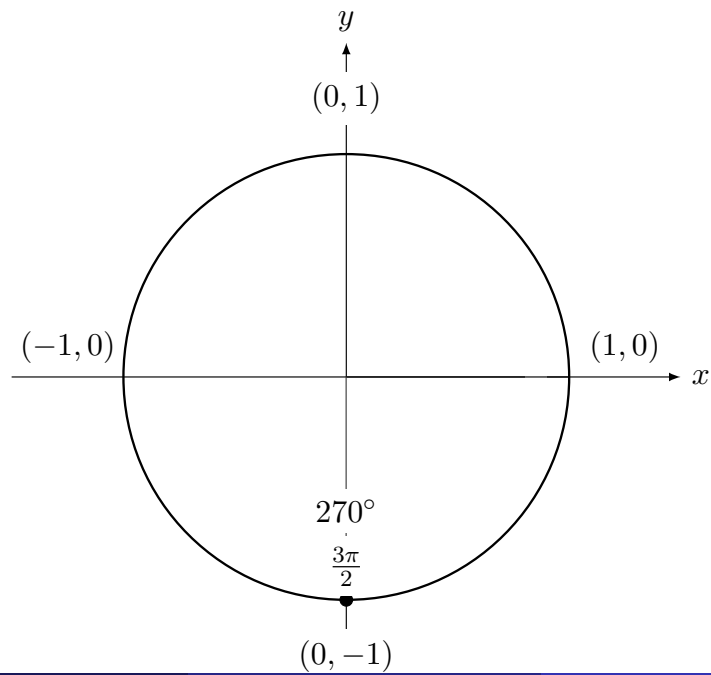
Let’s work through some examples of angles where it is **easy** to compute the cosine and sine.



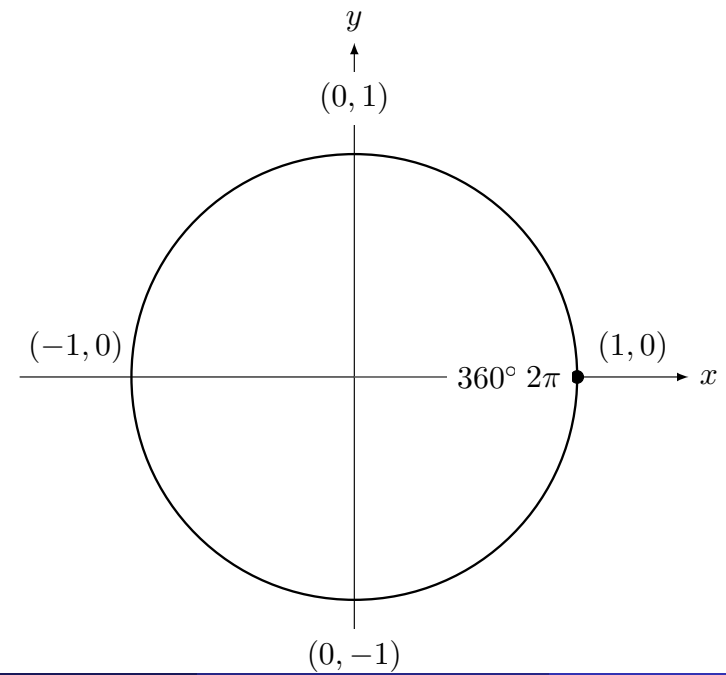




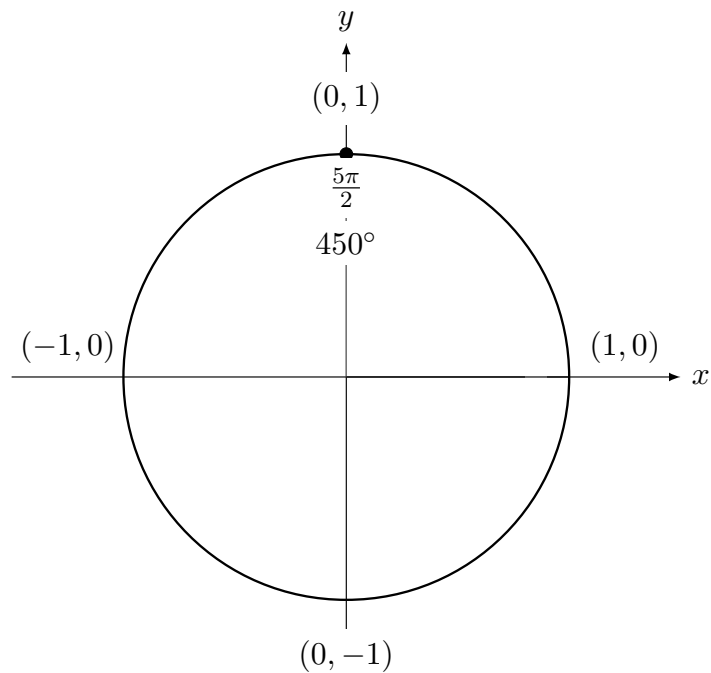
$$\cos \frac{3\pi}{2} = 0, \sin \frac{3\pi}{2} = -1$$



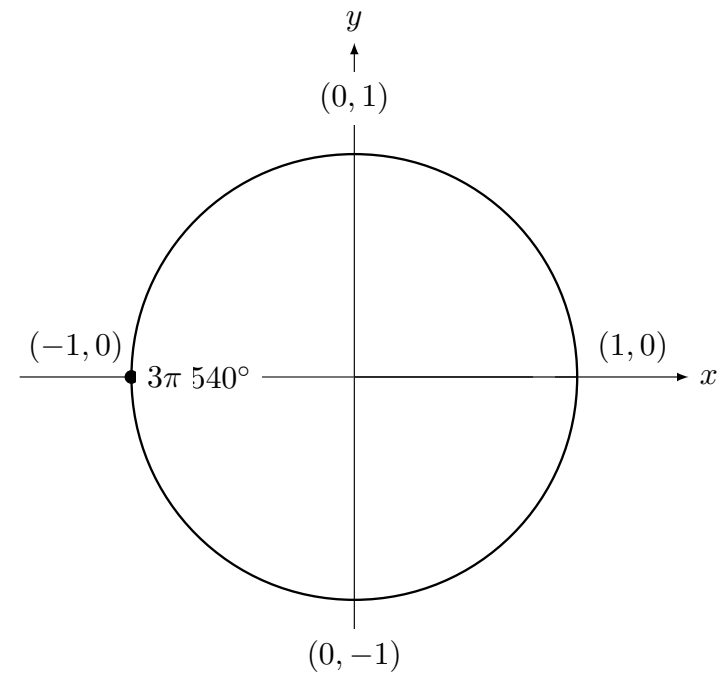
$$\cos 2\pi = 1, \sin 2\pi = 0$$



$$\cos \frac{5\pi}{2} = 0, \sin \frac{5\pi}{2} = 1$$

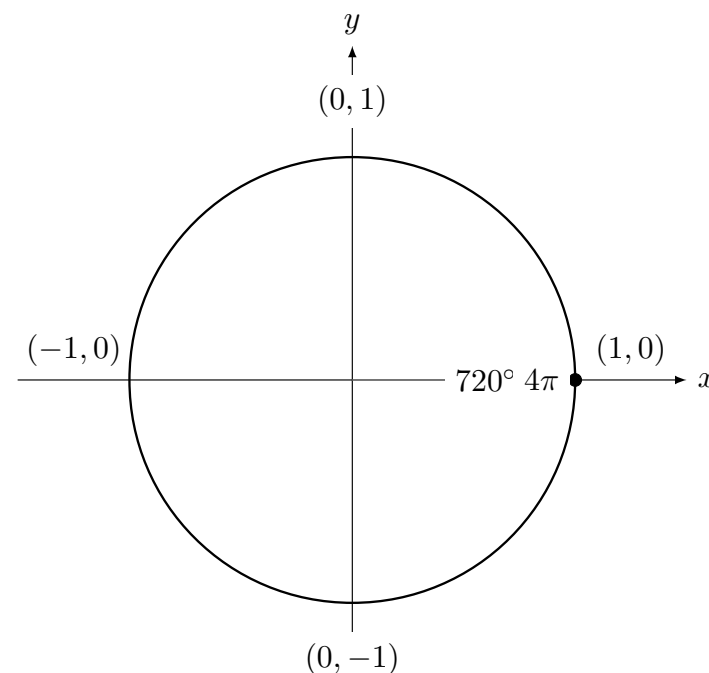
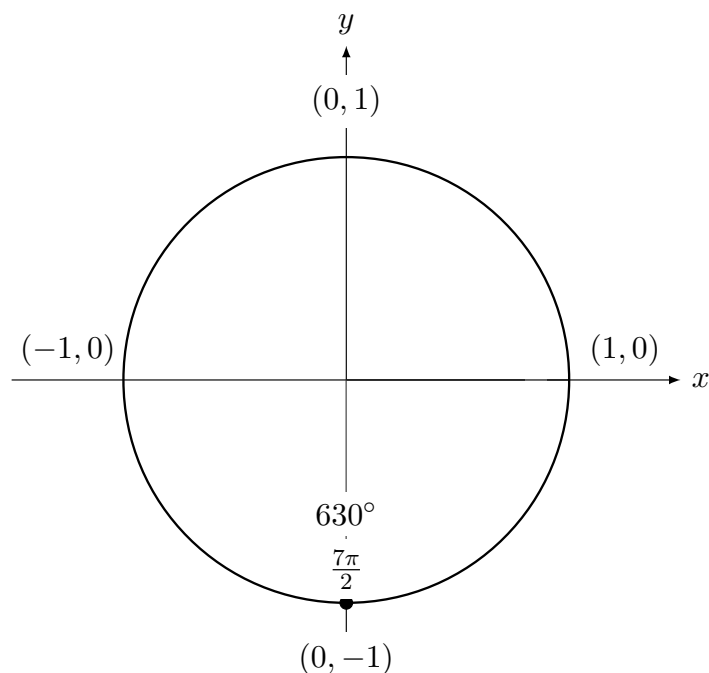


$$\cos 3\pi = -1, \sin 3\pi = 0$$



$$\cos \frac{7\pi}{2} = 0, \sin \frac{7\pi}{2} = -1$$

$$\cos 4\pi = 1, \sin 4\pi = 0$$



Some worked problems

Draw the unit circle and then find the exact value of cosine and sine for the following angles:

1 $\theta = \pi$.

Solution. π radians is 180 degrees, halfway around the circle. Start on the unit circle at (1, 0) and go halfway around (counterclockwise) to the point (-1, 0) at the far left side of the circle. The cosine is the x -value and the sine is the y -value. So $\cos(\pi) = -1$ and $\sin(\pi) = 0$.

Some worked problems

Draw the unit circle and then find the exact value of cosine and sine for the following angles:

2 $\theta = 3\pi$. **Solution.** Since π radians is 180 degrees, halfway around the circle, then the angle 3π is $1\frac{1}{2}$ times around the circle. Start on the unit circle at (1, 0) and go around once and then halfway around (counterclockwise) to the point (-1, 0). (Yes, this is the same point we would find if we only did π radians, so the answer will be the same as the previous problem.) $\cos(3\pi) = -1$ and $\sin(3\pi) = 0$.

Draw the unit circle and then find the exact value of cosine and sine for the following angles:

3 $\theta = \frac{7\pi}{2}$.

Solution. Since $\frac{7}{2}$ is equal to $3\frac{1}{2}$ then $\frac{7}{2}\pi$ radians ($= 630^\circ$) is an angle that goes around the circle one full revolution (360°) and then another 270 degrees (three-quarters of a revolution) to the point $(0, -1)$ at the bottom of the circle. So the answer is

$$\cos\left(\frac{7\pi}{2}\right) = 0 \text{ and } \sin\left(\frac{7\pi}{2}\right) = -1.$$

In the next presentation, we will look in depth at the *six* trig functions.

(End)