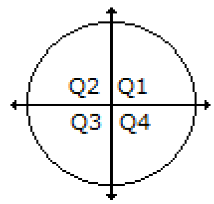
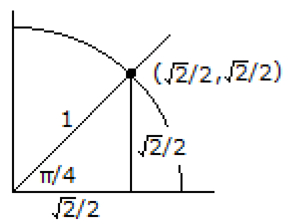
The Unit Circle
Exact Measurements and Symmetry

Consider the unit circle: a circle of radius 1, centered at the origin. The *x* and *y*-axes break up the plane into four quadrants, labeled 1-4, as shown below:



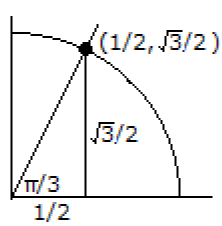
Now look at Quadrant 1. If we sketch in a ray at an angle of $\frac{\pi}{4}$ radians (45 degrees), we can calculate the coordinate (x, y) by using Pythagoras' Theorem to determine the lengths of the legs (remember, the *x*-coordinate is the length of the horizontal leg, the *y*-coordinate the length of the vertical leg):

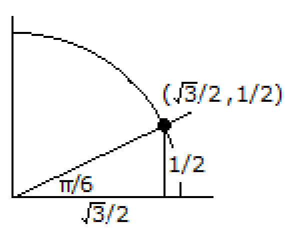


Therefore, we have these exact measurements:

- $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\tan\frac{\pi}{4} = 1$.

We can also sketch rays at angles of $\frac{\pi}{6}$ radians (30 degrees) and $\frac{\pi}{3}$ radians (60 degrees) in Quadrant 1. These form parts of an equilateral (all sides and angles equal) triangle. Using Pythagoras' Theorem, we can determine exact coordinates as shown below:





This gives us more exact measurements:

- $\sin\frac{\pi}{6} = \frac{1}{2}, \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$,
- $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \cos\frac{\pi}{3} = \frac{1}{2}$ and $\tan\frac{\pi}{3} = \sqrt{3}$.

You should memorize these so-called "First Quadrant Exact Measures". A neat way to memorize these values is shown on the next page:

- ***First Quadrant Exact Measures:***

1) Set up a table with three columns: angle measures for 0, 30, 45, 60 and 90 degrees and their equivalent radians, and a column for the cosine and sine values of these angles.
2) Fill in "blank roots over 2" in the cosine and sine columns.
3) Fill in these blank roots with 4, 3, 2, 1, 0 for the cosine, and 0, 1, 2, 3, 4 for the sine.
4) Simplify what you can, and you have the table!

θ	$\cos\theta$	$\sin\theta$
0	$\frac{\sqrt{\ }}{2}$	$\frac{\sqrt{\ }}{2}$
30 ($\frac{\pi}{6}$)	$\frac{\sqrt{\ }}{2}$	$\frac{\sqrt{\ }}{2}$
45 ($\frac{\pi}{4}$)	$\frac{\sqrt{\ }}{2}$	$\frac{\sqrt{\ }}{2}$
60 ($\frac{\pi}{3}$)	$\frac{\sqrt{\ }}{2}$	$\frac{\sqrt{\ }}{2}$
90 ($\frac{\pi}{2}$)	$\frac{\sqrt{\ }}{2}$	$\frac{\sqrt{\ }}{2}$

$\Rightarrow$

θ	$\cos\theta$	$\sin\theta$
0	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{0}}{2}$
30 ($\frac{\pi}{6}$)	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2}$
45 ($\frac{\pi}{4}$)	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60 ($\frac{\pi}{3}$)	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2}$
90 ($\frac{\pi}{2}$)	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{4}}{2}$

$\Rightarrow$

θ	$\cos\theta$	$\sin\theta$
0	$\frac{\sqrt{4}}{2}=1$	$\frac{\sqrt{0}}{2}=0$
30 ($\frac{\pi}{6}$)	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2}=\frac{1}{2}$
45 ($\frac{\pi}{4}$)	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60 ($\frac{\pi}{3}$)	$\frac{\sqrt{1}}{2}=\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
90 ($\frac{\pi}{2}$)	$\frac{\sqrt{0}}{2}=0$	$\frac{\sqrt{4}}{2}=1$

Other Quadrants:

The cosine, sine and tangent values change sign according to the quadrant in which the angle θ is located, according to this table:

Quadrant 2 $\cos\theta < 0$ $\sin\theta > 0$ $\tan\theta < 0$	Quadrant 1 $\cos\theta > 0$ $\sin\theta > 0$ $\tan\theta > 0$
Quadrant 3 $\cos\theta < 0$ $\sin\theta < 0$ $\tan\theta > 0$	Quadrant 4 $\cos\theta > 0$ $\sin\theta < 0$ $\tan\theta < 0$

To determine exact values in other quadrants, use symmetry to "revert" to a reference angle in the first quadrant, then attach a negative sign if necessary.

Examples:

- $\cos\frac{2\pi}{3} = -\frac{1}{2}$ since the angle $\frac{2\pi}{3}$ is in Quadrant 2, where the cosine is negative, and the angle $\frac{2\pi}{3}$ is symmetrical with the angle $\frac{\pi}{3}$ in Quadrant 1. So we look up $\cos\frac{\pi}{3}$ on our table, see that it is $\frac{1}{2}$, then attach the negative sign.
- $\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$. The angle $\frac{3\pi}{4}$ is in Quadrant 2, where the sine is positive. The angle $\frac{3\pi}{4}$ is symmaterical with the angle $\frac{\pi}{4}$ in Quadrant 1. We look up $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, and keep the sign positive.

Note: the tangent column can be found by remembering that $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

The Complete Unit Circle with Exact Measurements

Quadrant 1 θ	$\cos\theta$	$\sin\theta$		**Quadrant 3** θ	$\cos\theta$	$\sin\theta$
0 (0 deg)	1	0		π (180 deg)	–1	0
$\pi/6$ (30 deg)	$\sqrt{3}/2$	$1/2$		$7\pi/6$ (210 deg)	$-\sqrt{3}/2$	$-1/2$
$\pi/4$ (45 deg)	$\sqrt{2}/2$	$\sqrt{2}/2$		$5\pi/4$ (225 deg)	$-\sqrt{2}/2$	$-\sqrt{2}/2$
$\pi/3$ (60 deg)	$1/2$	$\sqrt{3}/2$		$4\pi/3$ (240 deg)	$-1/2$	$-\sqrt{3}/2$
Quadrant 2 θ				**Quadrant 4** θ		
$\pi/2$ (90 deg)	0	1		$3\pi/2$ (270 deg)	0	–1
$2\pi/3$ (120 deg)	$-1/2$	$\sqrt{3}/2$		$5\pi/3$ (300 deg)	$1/2$	$-\sqrt{3}/2$
$3\pi/4$ (135 deg)	$-\sqrt{2}/2$	$\sqrt{2}/2$		$7\pi/4$ (315 deg)	$\sqrt{2}/2$	$-\sqrt{2}/2$
$5\pi/6$ (150 deg)	$-\sqrt{3}/2$	$1/2$		$11\pi/6$ (330 deg)	$\sqrt{3}/2$	$-1/2$