

## Trigonometric Identities

### Basic Identities:

$$\sin \theta = \frac{y}{r} \quad (1)$$

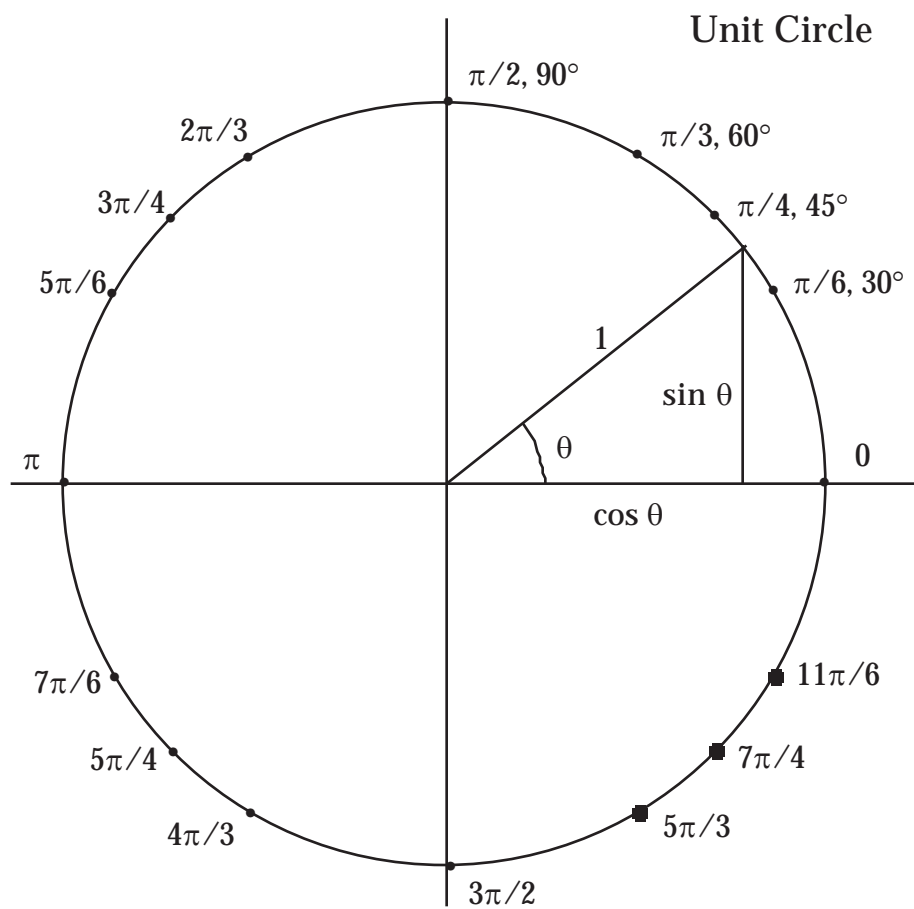
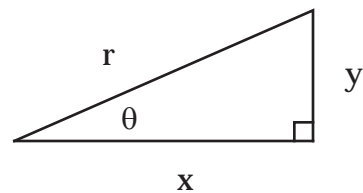
$$\cos \theta = \frac{x}{r} \quad (2)$$

$$\tan \theta = \frac{y}{x} = \frac{\sin \theta}{\cos \theta} \quad (3)$$

$$\cot \theta = \frac{x}{y} = \frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta} \quad (4)$$

$$\sec \theta = \frac{r}{x} = \frac{1}{\cos \theta} \quad (5)$$

$$\csc \theta = \frac{r}{y} = \frac{1}{\sin \theta} \quad (6)$$



**Table of Values from 0 to  $2\pi$ :**

degrees	0	30	45	60	90	120	135	150	180
radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	$\pi$
sin $\theta$ (y/r)	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos $\theta$ (x/r)	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	-1
tan $\theta$ (y/x)	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\rightarrow \pm \infty$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0
cot $\theta$ (x/y)	$\rightarrow \pm \infty$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	$\rightarrow \pm \infty$
sec $\theta$ (r/x)	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	$\rightarrow \pm \infty$	-2	$-\sqrt{2}$	$-\frac{2\sqrt{3}}{3}$	-1
csc $\theta$ (r/y)	$\rightarrow \pm \infty$	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	$\rightarrow \pm \infty$

**Table continued**

degrees	210	225	240	270	300	315	330	360
radians	$7\pi/6$	$5\pi/4$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$7\pi/4$	$11\pi/6$	$2\pi$
sin $\theta$ (y/r)	$\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos $\theta$ (x/r)	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
tan $\theta$ (y/x)	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\rightarrow \pm \infty$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0
cot $\theta$ (x/y)	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	$\rightarrow \pm \infty$
sec $\theta$ (r/x)	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	$\rightarrow \pm \infty$	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
csc $\theta$ (r/y)	-2	$-\sqrt{2}$	$-\frac{2\sqrt{3}}{3}$	-1	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	$\rightarrow \pm \infty$

**Some useful trigonometric relationships:**

$$\sin^2 \theta + \cos^2 \theta = 1 \quad (7)$$

$$1 + \tan^2 \theta = \sec^2 \theta \quad (\text{divide [ 7 ] by } \sin^2 \theta) \quad (8)$$

$$1 + \cot^2 \theta = \csc^2 \theta \quad (\text{divide [ 7 ] by } \cos^2 \theta) \quad (9)$$

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \quad (10)$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \quad (11)$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta} \quad (12)$$

$$\sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta) \quad (13)$$

$$\cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) \quad (14)$$

$$\sin 2\theta = 2 \sin \theta \cos \theta \quad (15)$$

$$\cos 2q = \cos^2 q - \sin^2 q = 2\cos^2 q - 1 = 1 - 2\sin^2 q \quad (16)$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} \quad (17)$$

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}} \quad (18)$$

$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}} \quad (19)$$

$$\tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta} \quad (20)$$

All of the above relationships are easily proved from Euler's identity

$$e^{i\theta} = \cos \theta + i \sin \theta, \quad (21a)$$

and it also follows that

$$e^{-i\theta} = \cos \theta - i \sin \theta, \quad (21b)$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos(-\theta) \quad (22)$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = -\sin(-\theta) \quad (23)$$

and these identities can be manipulated to get a new and sometimes more convenient expression for the trigonometric function of an angle. Just in case you doubt this method, we append some derivations:

$$\begin{aligned} \cos^2 \theta + \sin^2 \theta &= \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right)^2 + \left( \frac{e^{i\theta} - e^{-i\theta}}{2i} \right)^2 \\ &= \frac{e^{2i\theta} + 2 + e^{-2i\theta}}{4} + \frac{e^{2i\theta} - 2 + e^{-2i\theta}}{-4} \\ &= \frac{e^{2i\theta} + 2 + e^{-2i\theta} - e^{2i\theta} + 2 - e^{-2i\theta}}{4} \\ &= 1. \end{aligned} \quad (24)$$

$$\sin 2\theta = \frac{e^{2i\theta} - e^{-2i\theta}}{2i}$$

$$\begin{aligned}
&= \frac{(e^{i\theta})^2 - (e^{-i\theta})^2}{2i} \\
&= \frac{(e^{i\theta} + e^{-i\theta})(e^{i\theta} - e^{-i\theta})}{2i} \\
&= (e^{i\theta} + e^{-i\theta}) \sin \theta \\
&= 2 \cos \theta \sin \theta .
\end{aligned} \tag{25}$$

The hyperbolic functions are analogous to the trig functions and often arise in physical situations. Their relations to the trig functions are as follows:

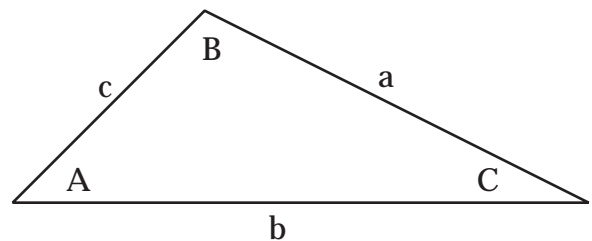
$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{-i(ix)} - e^{i(ix)}}{2} = -i \sin(ix) , \tag{26}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{-i(ix)} + e^{i(ix)}}{2} = \cos(ix) , \tag{27}$$

$$\cos x = \cosh \frac{x}{i} = \cosh(-ix) = \cosh(ix) , \tag{28}$$

$$\sin x = i \sinh \frac{x}{i} = i \sinh(-ix) = -i \sinh(ix) . \tag{29}$$

(The following laws apply for all triangles with angles, A, B and C and opposite side lengths as defined in the figure.)



$$\text{Law of Sines: } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{30}$$

$$\text{Law of Cosines: } c^2 = a^2 + b^2 - 2ab \cos C \tag{31}$$